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A recursive method for functionals of Poisson processes

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Functionals of Poisson processes arise in many statistical problems. They appear in problems involving heavy-tailed distributions in the study of limiting processes, while in Bayesian nonparametric statistics they are used as constructive representations for nonparametric priors. We describe a simple recursive method that is useful for characterizing Poisson process functionals that requires only the use of conditional probability. Applications of this technique to convex hulls, extremes, stable measures, infinitely divisible random variables and Bayesian nonparametric priors are discussed.

Keywords: convex hulls; Dirichlet process; extremes; gamma process; infinitely divisible random variables; point processes; stable processes

1. Introduction

Let $\{\Gamma_k\}$ be arrival times of a Poisson process with Lebesgue mean measure, independent of $\{Z_k\}$, which are independent and identically distributed (i.i.d.) random elements on some arbitrary measurable space $(\mathcal{X}, \mathcal{B})$. Thus $\Gamma_k = E_1 + \ldots + E_k$, where $\{E_k\}$ are i.i.d. standard Exp(1) random variables independent of $\{Z_k\}$. Functionals of the Poisson process

$$\Pi(\cdot) = \sum_{k=1}^{\infty} \varepsilon_{(Z_k, \Gamma_k)}(\cdot), \tag{1}$$

where ε_x is a discrete measure concentrated at x, appear frequently in the Bayesian nonparametric literature. For example, they are used to represent the Dirichlet process (Ferguson 1973; 1974) and the gamma process (Ferguson and Klass 1972; Kingman 1975) in a constructive form as an infinite sum representation. See Lo (1982) and Lo and Weng (1989) for applications of the gamma process in Bayesian nonparametric statistics, and Ferguson *et al.* (1992), Escobar and West (1998) and MacEachern (1998) for applications of the Dirichlet process. Applications of constructive sum representations for the Dirichlet process can be found in Ishwaran and James (2001).

Functionals of Π, and closely related expressions, also appear routinely in problems involving heavy-tailed distributions. Suppose X, X_1, $X_2, \ldots$ are i.i.d. random variables such that

$$P\{|X| > x\} = x^{-\alpha} L(x), \tag{2}$$

for some slowly varying function L at ∞ with $\alpha > 0$, and

$$\frac{P\{X > x\}}{P\{|X| > x\}} \to p \in [0, 1], \qquad \text{as } x \to \infty. \tag{3}$$

Let '$\Rightarrow$' denote weak convergence with respect to some appropriate topology. Under assumptions (2) and (3), there exists a sequence of non-negative constants $a_n \to \infty$ such that

$$\sum_{k=1}^{n} \varepsilon_{(k/n, a_n^{-1} X_k)}(\cdot) \Rightarrow \sum_{k=1}^{\infty} \varepsilon_{(Z_k, \delta_k \Gamma_k^{-1/\alpha})}(\cdot), \tag{4}$$

where $\{\delta_k\}$ are i.i.d. binary variables such that $P\{\delta_k = 1\} = p = 1 - P\{\delta_k = -1\}$, Z_k are i.i.d. Uniform[0, 1] random variables and $\{\Gamma_k, \delta_k, Z_k\}$ are mutually independent. See, for example, Davis and Resnick (1985a; 1985b) or Resnick (1986).

The limiting process in (4) is closely related to Π. It is a Poisson point process with mean measure $\mathrm{d}t \times \mathrm{d}\nu_\alpha$, where $\mathrm{d}t$ is Lebesgue measure on [0, 1] and ν_α is the Lévy measure defined by

$$\nu_\alpha(\mathrm{d}x) = \alpha(p x^{-\alpha-1} I\{0 < x < \infty\} + (1-p)|x|^{-\alpha-1} I\{-\infty < x < 0\})\,\mathrm{d}x. \tag{5}$$

This point process limit and its functionals often appear in the heavy-tail literature. For example, LePage *et al.* (1981) study this process in developing representations of stable random variables and stable processes. Davis and Resnick (1985a; 1985b), Resnick (1987, Chapter 4) and Mikosch *et al.* (1995) study (4) and its functionals in time series models with infinite-variance innovations, while de Haan (1984) used it for the representation of max-stable processes. The expected number of level crossings for a stationary, harmonizable, symmetric stable processes is described by Adler *et al.* (1993) as a functional of this point process. This limit process is also used by Rachev and Samorodnitsky (1993) in finance for option pricing formulae. A bootstrap application involving this process can be found in Knight (1989) and Zarepour (1999).

In this paper we introduce a new recursive method based on a simple conditioning technique that can be used for studying functionals of Poisson processes (see Embrechts *et al.* 1998 for other recursive methods). We use this approach to obtain characteristic functions and the distributions for some relatively complicated functionals. Despite the fact that this technique is not applicable in general, we find it a handy and useful method in many cases. Moreover, our approach is simple to use, requiring only knowledge of conditional probability.

We apply our recursion method to a variety of problems to illustrate its utility. In Section 2 we show how it can be used to establish the Ferguson and Klass (1972) almost sure sum representation for infinitely divisible random variables. We also give an elementary proof for the well-known stable law series representation on separable Hilbert spaces (LePage *et al.* 1981). Sections 3 and 4 look at the gamma process and the Dirichlet process, two widely used priors in Bayesian nonparametric statistics. There we apply our method to verify two well-known sum representations for these processes. Section 5 uses our recursion technique to find the distribution of the area of convex hulls from bivariate samples with coordinatewise regularly varying tails. We also look at the distribution for the maxima from bivariate heavy-tailed distributions.

2. Infinitely divisible random variables and stable laws

For our first illustration, we give an elementary proof of the Ferguson and Klass (1972) sum representation for infinitely divisible random variables. Specifically, we are interested in infinitely divisible non-Gaussian random variables with positive support whose characteristic functions can be expressed as

$$\phi(\theta) = \exp\left\{-\int_0^\infty (\exp(i\theta u) - 1)\,\mathrm{d}N(u)\right\}, \qquad -\infty < \theta < \infty, \tag{6}$$

where N (the Lévy measure) is a Borel measure defined on $(0, \infty)$ by $N(x) = \int_x^\infty \mathrm{d}N(u)$ and

$$\int_\epsilon^\infty N^{-1}(u)\,\mathrm{d}u < \infty, \qquad \text{for each } \epsilon > 0, \tag{7}$$

in which $N^{-1}(u) = \sup\{x : N(x) \leqslant u\}$. We further require that N is continuous. Thus, N is positive, continuous and non-increasing.

As shown in Ferguson and Klass (1972), an infinitely divisible random variable J with characteristic function (6) has an almost sure representation in terms of a homogeneous Poisson process:

Theorem 1. *Suppose that J has a characteristic function* (6) *with Lévy measure N as described above. Then J has the almost sure representation $J = \sum_{k=1}^\infty N^{-1}(\Gamma_k)$.*

The proof of the theorem is given in detail below, but it is worthwhile to give an informal description here as the same recursion method will be used repeatedly throughout the paper. A pattern that will emerge is that our technique generally involves conditioning on the value of Γ_1 in some kind of inifinte-term expression. Doing so allows us to work with a resulting recursive distributional equation, which we then set about solving by using the fact that Γ_1 has a standard Exp(1) distribution.

For example, to prove Theorem 1, we condition on the value of Γ_1 in the infinite sum

$$J(t) = \sum_{k=1}^\infty N^{-1}(\Gamma_k + t), \qquad t \geqslant 0,$$

and from this work out its characteristic function. Here, conditioning on the value $\Gamma_1 = y$ gives us the recursive distributional equation

$$(J(t)|\Gamma_1 = y) \overset{D}{=} N^{-1}(y + t) + J(t + y).$$

Using this to solve for the characteristic function of $J(t)$ and then setting $t=0$ will show that $J=J(0)$ has characteristic function (6), thus establishing the result.

Proof. First notice that the sequence $J(t)$ converges almost surely due to the integrability condition (7) and from the fact that N is decreasing. Thus, $J(t)$ is well defined. Conditioning on Γ_1, it follows from out discussion above that the characteristic function of $J(t)$ equals

$$\phi(\theta,\, t)=\mathrm{E}(\exp(\mathrm{i}\theta J(t)))=\int_0^\infty \exp(-y)\exp(\mathrm{i}\theta N^{-1}(y+t))\phi(\theta,\, t+y)\,\mathrm{d}y.$$

Use the change of variables $u=y+t$ and then multiply both sides by $\exp(-t)$. Differentiating both sides with respect to t, we obtain

$$\exp(-t)\left(\frac{\partial\phi(\theta,\, t)}{\partial t}-\phi(\theta,\, t)\right)=-\exp(-t+\mathrm{i}\theta N^{-1}(t))\,\phi(\theta,\, t),$$

which, rearranged, gives

$$\frac{\partial\phi(\theta,\, t)}{\partial t}=(1-\exp(\mathrm{i}\theta N^{-1}(t)))\,\phi(\theta,\, t).$$

The unique solution to this differential equation is

$$\phi(\theta,\, t)=\exp\left\{-\int_0^{N^{-1}(t)}(\exp(\mathrm{i}\theta u)-1)\,\mathrm{d}N(u)\right\}.$$

Setting $t=0$ and using $N(\infty)=0$ gives (6). □

Another interesting application of our recursion method is a simple proof for establishing the series representation for a stable law on a Hilbert space (see LePage *et al.* 1981). Here we give a proof using a characteristic function argument. Let X, X_1, X_2, ... be a sequence of i.i.d. random elements with a regularly varying tail defined on a separable Hilbert space $\mathcal{X}$. That is, assume there exists a sequence of positive constants $a_n\to\infty$ such that

$$nP\left\{a_n^{-1}\|X\|>x,\, \frac{X}{\|X\|}\in\cdot\right\}\xrightarrow{v}x^{-\alpha}Q(\cdot),\qquad \alpha>0, \tag{8}$$

where Q is a probability measure on $\mathcal{S}=\{x\in\mathcal{X}:\|x\|=1\}$ and '$\xrightarrow{v}$' denotes vague convergence. The assumption of a regularly varying tail implies that $G(x)=P\{\|X\|>x\}$ must equal $x^{-\alpha}L(x)$ for some slowly varying function L at ∞. Moreover, Zarepour (1999) shows that if $0<\alpha<1$, or $0<\alpha<2$ and Q is a symmetric probability measure (i.e. $Q(B)=Q(-B)$ for each Borel set B), then

$$\sum_{k=1}^n \varepsilon_{a_n^{-1}X_k}(\cdot)\overset{D}{=}\sum_{k=1}^n \varepsilon_{a_n^{-1}G^{-1}(\Gamma_k/\Gamma_{n+1})S_k}(\cdot)\overset{\text{a.s.}}{\Rightarrow}\sum_{k=1}^\infty \varepsilon_{S_k\Gamma_k^{-1/\alpha}}(\cdot), \tag{9}$$

where $\{S_k\}$ are i.i.d. random elements with distribution Q. Here we will show that the functional $\sum_{k=1}^\infty S_k\Gamma_k^{-1/\alpha}$ of the above limiting Poisson process has a stable law for the case when $0<\alpha<2$ and Γ^* is symmetric. Our method of proof will be quite similar to the recursion method used in proving Theorem 1. Our proof will show that the above functional has the same characteristic function as that derived in Kuelbs (1973).

For $t \geqslant 0$ define

$$X(t) = \sum_{k=1}^{\infty} S_k(\Gamma_k + t)^{-1/\alpha}, \qquad 0 < \alpha < 2,$$

and let $\mu(A, t) = PX(t) \in A\}$ for each Borel set A. By conditioning on Γ_1 and S_1, deduce that

$$\begin{aligned}\mu(A, t) &= \int_S \int_0^{\infty} P\{X(t+u) \in A - s(t+u)^{-1/\alpha}\} \exp(-u)\, \mathrm{d}u\, \mathrm{d}Q(s) \\ &= \int_S \int_0^{\infty} \mu(A - s(t+u)^{-1/\alpha}, t+u) \exp(-u)\, \mathrm{d}u\, \mathrm{d}Q(s).\end{aligned}$$

Let $\theta \in \mathcal{X}$. The characteristic function for $X(t)$ is

$$\begin{aligned}\phi(\theta, t) &= \int_{\mathcal{X}} \exp(\mathrm{i}\langle\theta, x\rangle)\mu(\mathrm{d}x, t) \\ &= \int_S \int_0^{\infty} \int_{\mathcal{X}} \exp(\mathrm{i}\langle\theta, x\rangle)\, \mu(\mathrm{d}x - s(t+u)^{-1/\alpha}, t+u) \exp(-u)\, \mathrm{d}u\, \mathrm{d}Q(s) \\ &= \int_S \int_0^{\infty} \phi(\theta, t+u) \exp(\mathrm{i}\langle\theta, s(t+u)^{-1/\alpha}\rangle) \exp(-u)\, \mathrm{d}u\, \mathrm{d}Q(s) \\ &= \int_S \int_t^{\infty} \phi(\theta, w) \exp(\mathrm{i}\langle\theta, sw^{-1/\alpha}\rangle) \exp(-(w-t))\, \mathrm{d}w\, \mathrm{d}Q(s).\end{aligned}$$

The assumption of symmetry for Q implies that

$$\exp(-t)\phi(\theta, t) = \int_S \int_t^{\infty} \phi(\theta, w) \cos\langle\theta, sw^{-1/\alpha}\rangle \exp(-w)\, \mathrm{d}w\, \mathrm{d}Q(s).$$

Taking the derivative with respect to t and solving this differential equation, we have

$$\phi(\theta, t) = \exp\left\{ -\int_t^{\infty} \int_S (1 - \cos\langle\theta, sw^{-1/\alpha}\rangle)\, \mathrm{d}Q(s)\, \mathrm{d}w \right\}.$$

For $t = 0$, we obtain the characteristic function

$$\phi(\theta, 0) = \exp\left\{ -C\int_S |\langle\theta, s\rangle|^{\alpha} \mathrm{d}Q(s) \right\},$$

where (see, for example, Samorodonitsky and Taqqu 1994, Section 2.3)

$$C = \begin{cases} \cos(\pi\alpha/2)\Gamma(1-\alpha) & \text{if } 0 < \alpha < 1 \\ \pi(2-\alpha)/(2\alpha) & \text{if } \alpha = 1 \\ \cos(\pi\alpha/2)\Gamma(3-\alpha)/(\alpha^2 - \alpha) & \text{if } 1 \leqslant \alpha < 2, \end{cases}$$

and $\Gamma(\cdot)$ denotes the gamma function. This is the desired characteristic function, similar to the result in Kuelbs (1973).

3. Sum-representations for the gamma process

Ferguson and Klass (1972) provided a sum representation for the gamma process based on the Lévy measure of a Gamma(α) random variable (i.e. a random variable with a gamma distribution having a shape parameter α and scale parameter 1). The Lévy measure of such a variable is the Borel measurable function N defined by

$$N(x) = \alpha \int_x^\infty \exp(-u)u^{-1}\mathrm{d}u, \qquad \text{for } x > 0. \tag{10}$$

(Notice that N satisfies the conditions of Theorem 1.) Call $\mathcal{G}_\mu$ a gamma process over a measurable space $(\mathcal{X}, \mathcal{B})$ with shape μ (a finite measure) if (i) $\mathcal{G}_\mu(A)$ is a Gamma($\mu(A)$) random variable for each measurable set A, and (ii) for each measurable partition $A_1, \ldots, A_d$ of $\mathcal{X}$, $\mathcal{G}_\mu(A_k)$ are independent Gamma($\mu(A_k)$) random variables for $k = 1, \ldots, d$. It is well known that the gamma process can be defined over arbitrary measurable spaces (see Kingman 1975).

Ferguson and Klass (1972) showed that the gamma process can also be defined constructively. Although they only specifically considered the unit interval $\mathcal{X} = [0, 1]$, their representation holds for arbitrary measurable spaces. For the Lévy measure (10) with $\alpha = \mu(\mathcal{X})$, they showed

$$\mathcal{G}_\mu(\cdot) = \sum_{k=1}^\infty N^{-1}(\Gamma_k)\varepsilon_{Z_k}(\cdot), \tag{11}$$

where $\{Z_k\}$ are i.i.d. $H(\cdot) = \mu(\cdot)/\mu(\mathcal{X})$ over $(\mathcal{X}, \mathcal{B})$ independent of $\{\Gamma_k\}$. Thus $\mathcal{G}_\mu$ can be written as a functional of a Poisson process Π of the form (1).

A difficulty in working with the Ferguson and Klass representation (11) is that no closed-form solution exists for the inverse of (10). Bondesson (1982) developed another Poisson process construction which avoided the need to work with Lévy measures. There it was shown that

$$\mathcal{G}_\mu(\cdot) = \sum_{k=1}^\infty \exp\left(-\frac{\Gamma_k}{\alpha}\right) V_k \varepsilon_{Z_k}(\cdot), \tag{12}$$

where $\{V_k\}$ is a sequence of i.i.d. standard Exp(1) random variables and $\{Z_k\}$ are i.i.d. H. It is assumed that $\{\Gamma_k, V_k, Z_k\}$ are mutually independent.

Here we will give a simple proof of the Bondesson construction using our recursive conditioning technique (the same method can be used to prove (11)). Let

$$X(t, \cdot) = \sum_{k=1}^\infty \exp\left(-\frac{\Gamma_k + t}{\alpha}\right) V_k \varepsilon_{Z_k}(\cdot), \qquad t \geqslant 0.$$

To establish (12) we need to show that for an arbitrary partition $A_1, \ldots, A_d$, $X(0, A_k)$ are independent Gamma($\mu(A_k)$) random variables for $k = 1, \dots, d$. However, as $X(t, \cdot)$ is a pure jump process, independence follows automatically (Breiman 1992, Section 14.7), and thus it suffices to show that $X(0, A)$ is a Gamma($\mu(A)$) random variable for each set A. We show this by evaluating the moment generating function for $X(t, A)$.

By conditioning on Γ_1, notice that

$$(X(t, A)|\Gamma_1 = u) \stackrel{D}{=} \exp\left(-\frac{u+t}{\alpha}\right) V_1 I\{Z_1 \in A\} + X(t+u, A),$$

where on the right-hand side $X(t+u, A)$ is independent of V_1 and Z_1. From this it follows that the moment generating function $M(\theta, t, A) = \mathrm{E}(\exp(\theta X(t, A)))$ equals

$$H(A)\int_0^\infty \int_0^\infty \exp(-u-v)\exp\left\{\theta v \exp\left(-\frac{u+t}{\alpha}\right)\right\} M(\theta, t+u, A)\,\mathrm{d}u\,\mathrm{d}v$$
$$+ H(A^c)\int_0^\infty \exp(-u) M(\theta, t+u, A)\,\mathrm{d}u,$$

where $\theta < 1$ is selected to ensure all integrals are finite. Now use the change of variable $u + t = w$ to write

$$\begin{aligned}\exp(-t)M(\theta, t, A) &= H(A)\int_t^\infty \int_0^\infty \exp(-w-v)\exp\left\{\theta v \exp\left(-\frac{w}{\alpha}\right)\right\} M(\theta, w, A)\,\mathrm{d}v\,\mathrm{d}w \\ &\quad + H(A^c)\int_t^\infty \exp(-w) M(\theta, w, A)\,\mathrm{d}w \\ &= H(A)\int_t^\infty \exp(-w)\left(1 - \theta\exp\left(-\frac{w}{\alpha}\right)\right)^{-1} M(\theta, w, A)\,\mathrm{d}w \\ &\quad + H(A^c)\int_t^\infty \exp(-w) M(\theta, w, A)\,\mathrm{d}w.\end{aligned}$$

Differentiate both sides with respect to t to obtain

$$\frac{\partial M(\theta, t, A)}{\partial t} = M(\theta, t, A)\left[H(A)\left(1 - \left(1 - \theta\exp\left(-\frac{t}{\alpha}\right)\right)^{-1}\right)\right].$$

Therefore,

$$M(\theta, t, A) = \exp\left\{\theta H(A)\int_t^\infty \frac{\exp(-u/\alpha)}{1 - \theta\exp(-u/\alpha)}\,\mathrm{d}u\right\}.$$

Setting $t = 0$, we see that the moment generating function for $X(0, A)$ equals

$$M(\theta, 0, A) = \exp\left\{\theta H(A)\int_0^\infty \frac{\exp(-u/\alpha)}{1 - \theta\exp(-u/\alpha)}\,\mathrm{d}u\right\} = (1-\theta)^{-\mu(A)},$$

and the result follows from this.

4. Poisson process representations for the Dirichlet process

Call $\mathcal{P}_\mu$ a Dirichlet process (Ferguson, 1973; 1974) with parameter $\mu = \alpha H$ (a finite non-null measure), written $\mathcal{P}_\mu \sim \mathrm{DP}(\alpha H)$, if, for each measurable partition $A_1, \ldots, A_{d+1}$,

$$(\mathcal{P}_\mu(A_1), \ldots, \mathcal{P}_\mu(A_d)) \sim \mathrm{Dir}(\mu(A_1), \ldots, \mu(A_{d+1})),$$

where $\alpha > 0$ and H is a probability measure over an arbitary measurable space $(\mathcal{X}, \mathcal{B})$. Although primarily introduced as a stochastic process by Ferguson, the Dirichlet process can also be written as a sum construction expressible as a functional of a Poisson process. For example, as one can also define $\mathcal{P}_\mu$ in terms of a gamma process as $\mathcal{P}_\mu(\cdot) = \mathcal{G}_\mu(\cdot)/\mathcal{G}_\mu(\mathcal{X})$, it follows that $\mathcal{P}_\mu$ can be defined in terms of the Ferguson and Klass representation (11) for the gamma process discussed earlier. Indeed, Ferguson (1973) showed that

$$\mathcal{P}_\mu(\cdot) = \sum_{l=1}^{\infty} \frac{N^{-1}(\Gamma_k)}{\sum_{l=1}^{\infty} N^{-1}(\Gamma_l)} \varepsilon_{Z_k}(\cdot),$$

where N is the Lévy measure (10) for a Gamma(α) random variable and $\{Z_k\}$ are i.i.d. H (it is interesting to note that Ferguson established the above construction for a Dirichlet process over arbitrary measurable spaces by working only with a gamma process over the unit interval $\mathcal{X} = [0, 1]$). Yet another Poisson process functional representation of the Dirichlet process includes the stick-breaking construction of Sethuraman (1994); see also McCloskey (1965), Patil and Taillie (1977), Sethuraman and Tiwari (1982), Hoppe (1987), Donnelly and Joyce (1989), Perman *et al.* (1992) and Pitman (1996) who discuss this stick-breaking construction. Ishwaran and Zarepour (2002) give a general discussion of sum representations for the Dirichlet process. See also Freedman (1963) and Fabius (1964) for earlier discussions of the Dirichlet process in the context of tail-free measures.

In the rest of this section we will focus on the Sethuraman (1994) stick-breaking construction. Sethuraman (1994) showed that the Dirichlet process $\mathcal{P}_\mu \sim \mathrm{DP}(\alpha H)$ could be written as

$$\mathcal{P}_\mu(\cdot) = V_1 \varepsilon_{Z_1}(\cdot) + \sum_{k=2}^{\infty}((1 - V_1)(1 - V_2) \ldots (1 - V_{k-1})V_k)\varepsilon_{Z_k}(\cdot), \tag{13}$$

where $\{V_k\}$ are i.i.d. Beta($1, \alpha$) random variables, independent of $\{Z_k\}$, which are i.i.d. H. Notice that (13) has the equivalent representation

$$\mathcal{P}_\mu(\cdot) = \sum_{k=1}^{\infty}\left(\exp\left(-\frac{\Gamma_{k-1}}{\alpha}\right) - \exp\left(-\frac{\Gamma_k}{\alpha}\right)\right)\varepsilon_{Z_k}(\cdot), \qquad \Gamma_0 = 0, \tag{14}$$

which is a functional of a Poisson process of the form (1). The equivalence in distribution between (13) and (14) follows since

$$
\begin{aligned}
&\exp\left(-\frac{\Gamma_{k-1}}{\alpha}\right) - \exp\left(-\frac{\Gamma_k}{\alpha}\right) \\
&\quad = \exp\left(-\frac{E_1}{\alpha}\right)\exp\left(-\frac{E_2}{\alpha}\right) \cdots \exp\left(-\frac{E_{k-1}}{\alpha}\right)\left(1 - \exp\left(-\frac{E_k}{\alpha}\right)\right) \\
&\quad \stackrel{D}{=} (1 - V_1)(1 - V_2)\cdots(1 - V_{k-1})V_k,
\end{aligned}
$$

as $\exp(-E_1/\alpha) \stackrel{D}{=} \text{Beta}(\alpha, 1) \stackrel{D}{=} 1 - \text{Beta}(1, \alpha)$, where $\{E_k\}$ are i.i.d. Exp(1) random variables.

Lemma 1. *For each non-negative integer $r \geqslant 0$ and each $\gamma > 0$,*

$$
\sum_{j=0}^{r} \frac{\gamma^{(j)}}{j!} = \frac{\gamma^{(r+1)}}{\gamma r!},
$$

where $\gamma^{(0)} = 1$ and $\gamma^{(j)} = \gamma(\gamma + 1)\cdots(\gamma + j - 1)$ for $j \geqslant 1$.

A simple application of this lemma, in combination with our conditioning argument, will show that (14) is the DP(αH) process.

Theorem 2. *For each measurable partition $A_1, \ldots, A_{d+1}$,*

$$
\mathbf{X} = (X_1, \ldots, X_d) = (\mathcal{P}_\mu(A_1), \ldots, \mathcal{P}_\mu(A_d)) \sim \text{Dir}(\alpha a_1, \ldots, \alpha a_{d+1}),
$$

where $a_j = P\{Z_1 \in A_j\}$ for $j = 1, \ldots, d + 1$. (Notice that $\sum_{j=1}^{d+1} a_j = 1$.)

Proof. The theorem is proved if we can show that $M(r_1, \ldots, r_d) = \text{E}(X_1^{r_1} \cdots X_d^{r_d})$, the joint moments for $\mathbf{X}$, agree with the joint moments for a Dir($\alpha a_1, \ldots, \alpha a_{d+1}$) distribution (see Lo 1991, Lemma 1). First observe that $\mathcal{P}_\mu(\cdot)$ can be written recursively as

$$
\mathcal{P}_\mu(\cdot) \stackrel{D}{=} \left(1 - \exp\left(-\frac{\Gamma_1}{\alpha}\right)\right)\varepsilon_{Z_1}(\cdot) + \exp\left(-\frac{\Gamma_1}{\alpha}\right)\mathcal{P}_\mu(\cdot),
$$

where, on the right-hand side, $\mathcal{P}_\mu$ is independent of Γ_1 and Z_1.

Let $\boldsymbol{\delta} = (\varepsilon_{Z_1}(A_1), \ldots, \varepsilon_{Z_1}(A_d))$. Let $\mathbf{e}_{d+1} = (0, \ldots, 0)$ and, for $j = 1, \ldots, d$, write $\mathbf{e}_j$ for the d-dimensional vector with jth coordinate one and zero elsewhere. Conditioning on $\boldsymbol{\delta}$, it follows that

$$
(\mathbf{X}|\boldsymbol{\delta} = \mathbf{e}_j) \stackrel{D}{=} \left(1 - \exp\left(-\frac{\Gamma_1}{\alpha}\right)\right)\mathbf{e}_j + \exp\left(-\frac{\Gamma_1}{\alpha}\right)\mathbf{X}, \qquad \text{for } j = 1, \ldots, d + 1,
$$

where, on the right-hand side, $\mathbf{X}$ is independent of Γ_1. Therefore, by conditioning on $\boldsymbol{\delta}$, and using the independence between $\mathbf{X}$ and Γ_1, write $M(r_1, \ldots, r_d)$ as

$$\sum_{j=1}^{d+1} \alpha_j \mathrm{E}(X_1^{r_1} \cdots X_d^{r_d} | \boldsymbol{\delta} = \mathbf{e}_j)$$

$$= \sum_{j=1}^{d} \alpha_j \mathrm{E}\left[\left(\left(1 - \exp\left(-\frac{\Gamma_1}{\alpha} \right) \right) + \exp\left(-\frac{\Gamma_1}{\alpha} \right) X_j \right)^{r_j} \exp\left(-\frac{(r - r_j)\Gamma_1}{\alpha} \right) \frac{X_1^{r_1} \cdots X_d^{r_d}}{X_j^{r_j}} \right]$$

$$+ \alpha_{d+1} M(r_1, \ldots, r_d) \mathrm{E}\left(\exp\left(-\frac{r\Gamma_1}{\alpha} \right) \right),$$

where $r = r_1 + \ldots + r_d$.

Expand the binomial term. Then simplify the last term on the right-hand side, combining its value with the left-hand side. Remembering that $\mathbf{X}$ and Γ_1 are independent, deduce that

$$M(r_1, \ldots, r_d)\left(1 - \alpha_{d+1} \frac{\alpha}{\alpha + r} \right) = \sum_{j=1}^{d} \alpha_j \sum_{i=0}^{r_j} M(r_1, \ldots, r_{j-1}, i, r_{j+1}, \ldots, r_d) U_{i,j}, \tag{15}$$

where

$$U_{i,j} = \binom{r_j}{i} \mathrm{E}[V_1^{r_j - i}(1 - V_1)^{r - r_j + i}] = \frac{\alpha^{(r - r_j + i)} \alpha r_j!}{\alpha^{(r)}\, (\alpha + r) i!}.$$

Because $M(0, \ldots, 0) = 1$, there must be a unique solution to (15). In fact, this is

$$M(r_1, \ldots, r_{j-1}, i, r_{j+1}, \ldots, r_d)$$

$$= \frac{(\alpha\alpha_1)^{(r_1)} \cdots (\alpha\alpha_{j-1})^{(r_{j-1})} (\alpha\alpha_j)^{(i)} (\alpha\alpha_{j+1})^{(r_{j+1})} \cdots (\alpha\alpha_d)^{(r_d)}}{\alpha^{(r - r_j + i)}},$$

which is the $(r_1, \ldots, r_{j-1}, i, r_{j+1}, \ldots, r_d)$th joint moment for a $\mathrm{Dir}(\alpha\alpha_1, \ldots, \alpha\alpha_{d+1})$ distribution. To check that this is the correct solution, plug this value into the right-hand side of (15), and use Lemma 1 to obtain

$$\sum_{j=1}^{d} \alpha_j \frac{(\alpha\alpha_1)^{(r_1)} \cdots (\alpha\alpha_d)^{(r_d)} \alpha r_j!}{(\alpha\alpha_j)^{(r_j)} \alpha^{(r)} (\alpha + r)} \sum_{i=0}^{r_j} \frac{(\alpha\alpha_j)^{(i)}}{i!} = \frac{M(r_1, \ldots, r_d)}{\alpha + r} \sum_{j=1}^{d} (\alpha\alpha_j + r_j),$$

which can be seen to be equal to the left-hand side of (15). □

5. Joint distribution for sums and maxima

Let $\{X_k\}$ be an i.i.d. sequence of random variables such that there exist constants $a_n > 0$ and b_n such that $a_n^{-1} \sum_{k=1}^{n} X_k - b_n$ has a non-degenerate limiting distribution. Also assume there exist constants $c_n > 0$ and d_n such that $c_n^{-1} \bigvee_{k=1}^{n} X_k - d_n$ has a non-degenerate limiting distribution. Chow and Teugels (1979) show that these two limits are dependent if and only if the sum is in the domain of attraction of a stable law and the maximum is in the domain of attraction of a Fréchet distribution. Distributional results for the joint distribution of the sum and maximum in the dependent case are generally hard to come by. Here we focus on this case using our recursive technique to study functionals from the resulting joint distribution.

Suppose, then, that the $\{X_k\}$ satisfy conditions (2) and (3) for $0 < \alpha < 1$, or that they satisfy these conditions with $0 < \alpha < 2$ and $p = \frac{1}{2}$. Then there exists a positive sequence $a_n \to \infty$ and δ_k as before such that

$$\left(a_n^{-1} \sum_{k=1}^{n} X_k, \, a_n^{-1} \bigvee_{k=1}^{n} X_k \right) \Rightarrow \left(\sum_{k=1}^{\infty} \delta_k \Gamma_k^{-1/\alpha}, \, \bigvee_{k=1}^{\infty} \delta_k \Gamma_k^{-1/\alpha} \right).$$

We will consider functions of the right-hand side, which are functionals of a Poisson process of the form (4). In fact, we will study a more general problem by considering functions of the bivariate vector

$$(X(t), \, Y(t)) = \left(\sum_{k=1}^{\infty} \frac{U_k}{(\Gamma_k + t)^{1/\alpha}}, \, \bigvee_{k=1}^{\infty} \frac{V_k}{(\Gamma_k + t)^{1/\alpha}} \right), \qquad t \geqslant 0,$$

where $\{(U_k, \, V_k)\}$ are i.i.d. and are assumed to be independent of $\{\Gamma_k\}$. Write $F(\cdot)$ and $G(\cdot)$ for the distribution of U_1 and V_1 respectively, and let $G(\cdot | U_1 = u)$ denote the conditional distribution of V_1 given U_1 (we allow U_1 to be conditionally dependent on V_1). In what follows we shall assume that the series $X(t)$ is convergent (conditions ensuring convergence will be developed shortly).

As our first step, we obtain a representation for

$$\phi(\theta, \, y, \, t) = \mathrm{E}[\exp(\mathrm{i}\theta X(t)) I\{Y(t) \leqslant y\}], \qquad -\infty < \theta, \, y < \infty,$$

which can help in characterizing certain Poisson process functionals. For example, later in Section 5.2 these calculations will help in finding the limiting distribution of the area of a convex hull from a bivariate sample with coordinatewise regularly varying tails. We will also be able to work out the distribution for the bivariate maximum from a heavy-tailed distribution (Section 5.1).

By conditioning on Γ_1 and $(U_1, \, V_1)$, we obtain

$$\begin{aligned} \phi(\theta, \, y, \, t) &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^{y(z+t)^{1/\alpha}} \exp\{-z + \mathrm{i}\theta u (z+t)^{-1/\alpha}\} \phi(\theta, \, y, \, t+z) \, \mathrm{d}G(v|u) \, \mathrm{d}F(u) \, \mathrm{d}z \\ &= \int_0^\infty \int_{-\infty}^\infty \exp\{-z + \mathrm{i}\theta u (z+t)^{-1/\alpha}\} \phi(\theta, \, y, \, z+t) \, G(y(z+t)^{1/\alpha} | u) \, \mathrm{d}F(u) \, \mathrm{d}z. \end{aligned}$$

Use the change of variable $s = z + t$ to obtain

$$\phi(\theta, \, y, \, t) = \int_t^\infty \int_{-\infty}^\infty \exp(-s + t) \exp(\mathrm{i}\theta u s^{-1/\alpha}) \, \phi(\theta, \, y, \, s) G(y s^{1/\alpha} | u) \, \mathrm{d}F(u) \, \mathrm{d}s.$$

Thus,

$$\frac{\partial \phi(\theta, \, y, \, t)}{\partial t} = \phi(\theta, \, y, \, t) \left[1 - \int_{-\infty}^\infty \exp(\mathrm{i}\theta u t^{-1/\alpha}) \, G(y t^{1/\alpha} | u) \, \mathrm{d}F(u) \right].$$

The solution to this differential equation is

$$\phi(\theta, y, t) = \exp\left\{-\int_t^{\infty}\left[1 - \int_{-\infty}^{\infty}\exp(\mathrm{i}\theta u z^{-1/\alpha})\, G(yz^{1/\alpha}|u)\,\mathrm{d}F(u)\right]\mathrm{d}z\right\}.$$

Thus, by taking $\theta = 0$, we have

$$P\left\{\bigvee_{k=1}^{\infty}\frac{V_k}{(\Gamma_k + t)^{1/\alpha}} \leqslant y\right\} = \exp\left\{-\int_t^{\infty}[1 - G(yz^{1/\alpha})]\,\mathrm{d}z\right\}. \tag{16}$$

In particular, if $V_k = \delta_k$,

$$P\left\{\bigvee_{k=1}^{\infty}\frac{\delta_k}{(\Gamma_k + t)^{1/\alpha}} \leqslant y\right\} = \exp\{-py^{-\alpha} + pt\}I\{y \in (0, t^{-1/\alpha})\} + I\{y \in [t^{-1/\alpha}, \infty)\}. \tag{17}$$

Also, letting $y \to \infty$,

$$\mathrm{E}[\exp(\mathrm{i}\theta X(0))] = \exp\left\{-\int_0^{\infty}[1 - \mathrm{E}(\exp(\mathrm{i}\theta U_1 z^{-1/\alpha}))]\,\mathrm{d}z\right\}.$$

Now we see that $X(t)$ converges if and only if the integral in the exponent of the above expression is finite. This holds, for example, if $\{U_k\}$ are symmetric or $0 < \alpha < 1$.

5.1. Applications to bivariate maxima

Expression (16) above can be used to derive the limiting distribution for the coordinatewise maximum of a bivariate sample from a heavy-tailed distribution. Suppose $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \ldots$ are i.i.d. random vectors on $\mathbb{R}^2$ with a regularly varying tail. That is, suppose there exist positive constants $a_n \to \infty$ so that (8) holds (with respect to Euclidean norm). Equivalently,

$$\sum_{k=1}^{n}\varepsilon_{a_n^{-1}\mathbf{X}_k}(\cdot) \Rightarrow \sum_{k=1}^{\infty}\varepsilon_{\mathbf{J}_k}(\cdot),$$

where the right-hand side is a Poisson point process similar to the limit given in (9). In particular, $\mathbf{J}_k = \Gamma_k^{-1/\alpha}(\cos\Theta_k, \sin\Theta_k)$, where $\{\Theta_k\}$ is an i.i.d. sequence taking values in $[0, 2\pi]$.

The limiting bivariate maximum is the functional

$$\left(\bigvee_{k=1}^{\infty}\frac{\cos\Theta_k}{\Gamma_k^{1/\alpha}}, \bigvee_{k=1}^{\infty}\frac{\sin\Theta_k}{\Gamma_k^{1/\alpha}}\right)$$

of the above Poisson process. Without loss of generality, we can assume $\alpha = 1$. To find its distribution, let $x > 0$, $y > 0$ and consider

$$V_k = \frac{\cos\Theta_k}{x}\bigvee\frac{\sin\Theta_k}{y}, \qquad k = 1, 2, \ldots.$$

Let

$$K(x, y) = P\left\{\bigvee_{k=1}^{\infty} \frac{\cos \Theta_k}{\Gamma_k} \leqslant x, \bigvee_{k=1}^{\infty} \frac{\sin \Theta_k}{\Gamma_k} \leqslant y\right\} = P\left\{\bigvee_{k=1}^{\infty} \frac{V_k}{\Gamma_k} \leqslant 1\right\}.$$

Therefore by (16),

$$K(x, y) = \exp\left\{-\int_0^{\infty} (1 - G(u))\, \mathrm{d}u\right\},$$

where G is the distribution function for $\{V_k\}$. For example, if $\{\Theta_k\}$ are uniformly distributed on $[0, 2\pi]$ (the bivariate Cauchy case) we obtain

$$\int_0^{\infty} (1 - G(u))\, \mathrm{d}u = \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} I\left\{\frac{\cos\theta}{x} \bigvee \frac{\sin\theta}{y} \geqslant u\right\} \mathrm{d}\theta\, \mathrm{d}u$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\cos\theta}{x} \bigvee \frac{\sin\theta}{y}\right) \mathrm{d}\theta.$$

Simple calculations now show that

$$K(x, y) = \exp\left\{-\frac{1}{2\pi}(x^{-1} + y^{-1} + (x^2 + y^2)^{-1/2})\right\}.$$

5.2. Applications to bivariate convex hulls

The convex hull of observations in $\mathbb{R}^d$, the smallest closed convex set containing the observations, is an alternative statistic to extremes in multivariate observations. If $d = 1$ the convex hull of the random sample $X_1, \ldots, X_n$ is the closed random interval $[\bigwedge_{k=1}^n X_k, \bigvee_{k=1}^n X_k]$.

Write Conv for the convex hull. By using a continuity argument, Davis *et al.* (1987) show that if $\mathbf{X}_k$ are random elements in $\mathbb{R}^2$ with a regularly varying tail, then

$$\mathrm{Conv}\{a_n^{-1}\mathbf{X}_1, \ldots, a_n^{-1}\mathbf{X}_n\} \Rightarrow \mathrm{Conv}\{\mathbf{J}_1, \mathbf{J}_2, \ldots\},$$

where convergence is with respect to the Hausdorff topology (see Matheron 1975), and $\{\mathbf{J}_k\}$ are points of a Poisson point process with mean measure ν. The asymptotic behaviour of functions such as Area (the area of a convex hull), Peri (the perimeter of a convex hull) and Vert (the number of vertices of a convex hull) were also studied thoroughly in Davis *et al.* (1987). They showed that (for $d = 2$)

$$\mathrm{Area}(\mathrm{Conv}\{a_n^{-1}\mathbf{X}_1, \ldots, a_n^{-1}\mathbf{X}_n\}) \Rightarrow \mathrm{Area}(\mathrm{Conv}\{\mathbf{J}_1, \mathbf{J}_2, \ldots\}).$$

Analogous convergence results were shown to hold for Peri, as well as for Vert.

In general, the limiting distribution of functionals such as the area and perimeter of a convex hull can be expressed in terms of $\{\mathbf{J}_k\}$, but their distributions remain intractable. Moreover, direct simulations from such distributions are not feasible in general. Zarepour (1999) suggests a bootstrap technique for evaluating distributions. There, under a specific resampling plan, it is shown that the bootstrap can be used to consistently estimate distributions for functionals such as Area and Peri.

However, under specific choices for the mean measure ν, explicit representations for distributions of functionals are possible using our recursion method. We will consider the case $d = 2$. Assume that $\{(X_k, Y_k)\}$ is an i.i.d. sequence of random vectors on $\mathbb{R}^2$ such that there exist non-negative sequences of constants $a_n \to \infty$ and $b_n \to \infty$ such that

$$nP\{(a_n^{-1}X_k, b_n^{-1}Y_k) \in (\mathrm{d}x, \mathrm{d}y)\} \xrightarrow{v} \nu_\alpha(\mathrm{d}x)\varepsilon_0(\mathrm{d}y) + \varepsilon_0(\mathrm{d}x)\nu_\beta(\mathrm{d}y), \tag{18}$$

where ν_α and ν_β are Lévy measures of the form (5). For example, such a limit holds if $\{X_k\}$ and $\{Y_k\}$ are independent with each sequence satisfying conditions (2) and (3) for exponents α and β respectively (note that the ps need not be the same). However, in general, condition (18) only requires independence in the limit.

Under (18), we have

$$\sum_{k=1}^{n} \varepsilon_{(a_n^{-1}X_k, b_n^{-1}Y_k)}(\cdot) \Rightarrow \sum_{k=1}^{\infty} \varepsilon_{(J_k^{(1)},0)}(\cdot) + \sum_{k=1}^{\infty} \varepsilon_{(0,J_k^{(2)})}(\cdot),$$

where $\sum_{k=1}^{\infty} \varepsilon_{(J_k^{(1)},0)}(\cdot)$ and $\sum_{k=1}^{\infty} \varepsilon_{(0,J_k^{(2)})}(\cdot)$ are independent Poisson point processes with mean measures ν_α and ν_β, respectively. Thus, if $0 < p < 1$,

$$\begin{aligned}&\mathrm{Conv}\{(a_n^{-1}X_1, b_n^{-1}Y_1), \ldots, (a_n^{-1}X_n, b_n^{-1}Y_n)\}\\ &\Rightarrow \mathrm{Conv}\left\{\left(\bigvee_{k=1}^{\infty} J_k^{(1)}, 0\right), \left(\bigwedge_{k=1}^{\infty} J_k^{(1)}, 0\right), \left(0, \bigvee_{k=1}^{\infty} J_k^{(2)}\right), \left(0, \bigwedge_{k=1}^{\infty} J_k^{(2)}\right)\right\}\end{aligned}$$

and

$$\mathrm{Vert}\{(a_n^{-1}X_1, b_n^{-1}Y_1), \ldots, (a_n^{-1}X_n, b_n^{-1}Y_n)\} \xRightarrow{p} 4.$$

Moreover,

$$\mathrm{Area}(\mathrm{Conv}\{(a_n^{-1}X_1, b_n^{-1}Y_1), \ldots, (a_n^{-1}X_n, b_n^{-1}Y_n)\}) \Rightarrow \frac{R_\infty(X)R_\infty(Y)}{2},$$

where $R_\infty(X) = \bigvee_{k=1}^{\infty} J_k^{(1)} - \bigwedge_{k=1}^{\infty} J_k^{(1)}$ and $R_\infty(Y) = \bigvee_{k=1}^{\infty} J_k^{(2)} - \bigwedge_{k=1}^{\infty} J_k^{(2)}$. Thus, the convex hull for observations from a heavy-tailed distribution corresponds to the extreme points from each coordinate. Consequently, such a region would then have only four vertex points (for some empirical evidence of this, see Figure 1 which plots the convex hull for a bivariate sample from a Cauchy distribution).

We can work out the distribution for Area using our recursive method and our earlier derivation (17). First observe that $R_\infty(X) = \bigvee_{k=1}^{\infty} \delta_k\Gamma_k^{-1/\alpha} - \bigwedge_{k=1}^{\infty} \delta_k\Gamma_k^{-1/\alpha}$. A similar representation holds for $R_\infty(Y)$ (the two terms are independent). Conditioning on δ_1 and Γ_1, and then using (17), deduce that

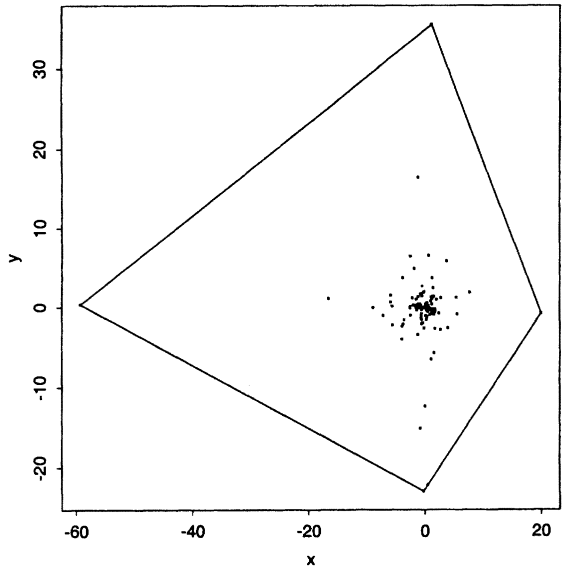

Figure 1. The convex hull from 1000 observations of a bivariate Cauchy distribution with independent components (observations indicated by points). Notice that the diameters of the convex hull are near perpendicular.

$$
\begin{aligned}
P\{R_\infty(X) \leqslant x\} &= P\left\{\bigvee_{k=1}^{\infty} \delta_k \Gamma_k^{-1/\alpha} + \bigvee_{k=1}^{\infty} -\delta_k \Gamma_k^{-1/\alpha} \leqslant x\right\} \\
&= p\int_0^\infty P\left\{\bigvee_{k=1}^{\infty} -\delta_k(\Gamma_k + y)^{-1/\alpha} \leqslant x - y^{-1/\alpha}\right\}\exp(-y)\,\mathrm{d}y \\
&\quad + (1-p)\int_0^\infty P\left\{\bigvee_{k=1}^{\infty} \delta_k(\Gamma_k + y)^{-1/\alpha} \leqslant x - y^{-1/\alpha}\right\}\exp(-y)\,\mathrm{d}y \\
&= \exp\left\{-\left(\frac{x}{2}\right)^{-\alpha}\right\} + p\int_0^{(x/2)^{-\alpha}} \exp\{-p(x - y^{-1/\alpha})^{-\alpha} - (1-p)y\}\,\mathrm{d}y \\
&\quad + (1-p)\int_0^{(x/2)^{-\alpha}} \exp\{-(1-p)(x - y^{-1/\alpha})^{-\alpha} - py\}\,\mathrm{d}y.
\end{aligned}
$$

A similar result holds for $P\{R_\infty(Y) \leqslant y\}$.

When $p = 1$ (the case where the observations are positive), it follows that

$$\mathrm{Vert}\{(a_n^{-1}X_1, b_n^{-1}Y_1), \ldots, (a_n^{-1}X_n, b_n^{-1}Y_n)\} \overset{P}{\Rightarrow} 3,$$

and from an argument similar to that above that

$$\text{Area}(\text{Conv}\{(a_n^{-1}X_1, b_n^{-1}Y_1), \ldots, (a_n^{-1}X_n, b_n^{-1}Y_n)\}) \Rightarrow E_1^{-1/\alpha}E_2^{-1/\beta}/2,$$

where E_1 and E_2 are independent standard Exp(1) random variables.

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