SERIES REPRESENTATIONS FOR MULTIVARIATE GENERALIZED GAMMA PROCESSES VIA A SCALE INVARIANCE PRINCIPLE

Hemant Ishwaran and Mahmoud Zarepour

Cleveland Clinic and University of Ottawa

Abstract: We introduce a scale invariance property for Poisson point processes and use this property to define a series representation for a correlated bivariate gamma process. This approach is quite general and can be used to define other types of multidimensional Lévy processes with given marginals. Some important special cases are bivariate G-processes, bivariate variance gamma processes and multivariate Dirichlet processes. Using the scale invariance principle we show how to construct simple approximations to these multivariate processes.

Key words and phrases: Correlated process, easure, G-measure.

1. Introduction.

The univariate gamma process, and its various extensions, plays an important role in a variety of applications. For example, the Dirichlet process, a fundamental prior used in a wide array of Bayesian nonparametric problems, is defined in terms of a gamma process (Ferguson and Klass (1972) and Ferguson (1973)). This construction is often expressed as an infinite series representation, and is defined as follows in notation that will be used throughout the paper. Let $\{X_i\}$ be a sequence of i.i.d. random variables with distribution P_0 over a general probability space $(\mathscr{X}, \mathscr{A})$. Let $\Gamma_n = \sum_{i=1}^{n} E_i$, where $\{E_i\}$ is a sequence of i.i.d. $\exp(1)$ random variables, constructed independently of $\{X_i\}$. The Dirichlet process with parameter αP_0 is defined as

$$\mathscr{P}(\cdot) = \frac{G_{\alpha,1}(\cdot)}{G_{\alpha,1}(\mathscr{X})},$$

where $G_{\alpha,\beta}$ is a gamma process, expressible as

$$G_{\alpha,\beta}(\cdot) = \sum_{i=1}^{\infty} N_{\alpha,\beta}^{-1}(\Gamma_i)\varepsilon_{X_i}(\cdot), \tag{1.1}$$

where $N_{\alpha,\beta}(x) = \alpha \int_x^\infty u^{-1} \exp(-u/\beta)\, du$, $x > 0$, is the Lévy measure for a gamma random variable with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$. Here $\varepsilon_X(\cdot)$ is a discrete measure concentrated on X.

The weighted gamma process (Lo (1982) and Lo and Weng (1989)), an extension of the gamma process (1.1), is another popular prior used in Bayesian noparametric inference. In this extension, the scale parameter β is replaced with an arbitrary positive function $\beta(\cdot)$. This is useful because it leads to a type of conjugacy for multiplicative intensity models (Lo and Weng (1989)). Applications of the weighted gamma process originally included the life-testing model, the multiple decrement model, birth and death processes and branching processes (Lo and Weng (1989)). See also Dykstra and Laud (1981). Extensions to a more general class of multiplicative intensity models as well as spatial processes were considered in Ishwaran and James (2004). See also James (2003).

Gamma processes also appear in finance. The widely studied variance gamma process (Madan, Carr and Chang (1998)) is defined as

$$\log p(t) = m G_{\alpha,1}(t) + \sigma W(G_{\alpha,1}(t)),$$

where $p(t)$ is the price of a stock at time t, and $W(\cdot)$ is standard Brownian motion independent of $G_{\alpha,1}(\cdot)$. The variance gamma process builds on the idea of modeling stock prices using a subordinated stochastic process. Importantly, however, the use of the gamma process as the directing process of the subordinator is different from traditional methods that have relied on infinite variance processes. See Madan, Carr and Chang (1998), Mandelbrot (1963), Fama (1965) and Mandelbrot and Taylor (1967).

A three-parameter distribution $\mathscr{G}(\alpha, \delta, \theta)$ for positive random variables was introduced in Hougaard (1986) and Aalen (1992), characterized by the Laplace transform

$$L_{\alpha,\delta,\theta}(s) = \exp\left(-\frac{\delta}{\alpha}\Big((\theta+s)^\alpha - \theta^\alpha\Big)\right), \quad s \geq 0, \tag{1.2}$$

where α, δ and θ satisfy

$$(\alpha, \delta, \theta) \in (0,1] \times (0,\infty) \times [0,\infty) \cup (-\infty,0] \times (0,\infty) \times (0,\infty). \tag{1.3}$$

This family was later used by Brix (1999) to define another important extension of the gamma process, the so-called G-measure. As in Brix (1999), call δ the shape parameter, α the index parameter, and θ the intensity parameter. Let $\kappa(\cdot)$ be a bounded positive measure on $(\mathscr{X}, \mathscr{A})$. A random measure $\nu(\cdot)$ is said to be a $\mathscr{G}(\alpha, \kappa, \theta)$ measure, or simply a G-measure, if (i) $\nu(A) \sim \mathscr{G}(\alpha, \kappa(A), \theta)$ for any measurable set A, and (ii) $\nu(\cdot)$ has independent increments. The gamma process is a special case of a G-measure and is obtained by taking the limit $\alpha \to 0$. See Brix (1999) for more background on G-measures and for discussion on their many useful applications.

1.1. Multivariate extensions

Given the wide applicability of the gamma process, it is of interest to extend the process to a multivariate setting. One way to approach this problem is through the use of copula functions (Sklar (1959)). However, while this is theoretically viable, copulas pose practical problems as there is no straightforward method for choosing the right class of them. Other disadvantages of this technique are described in Cont and Tankov (2004) and Kallsen and Tankov (2006). In Cont and Tankov (2004) and Kallsen and Tankov (2006), an alternate method using what was termed a Lévy copula was introduced. This elegant new technique has many advantages. Lévy copulas are extremely flexible and give rise to all multivariate Lévy processes with given marginals. Using Lévy copulas ensures that the resulting multivariate process is Lévy, and therefore is infinitely divisible.

In this paper we introduce a scale invariance principle for Poisson processes. As we show, this principle has wide reaching applications to multivariate processes. In particular, in Section 3 we show how the principle can be used to define a bivariate (and multivariate) gamma process simply, thus adding a new method for constructing such processes to the literature. A useful property of the bivariate process is that it is expressed as an infinite series representation with the specific choice of random variables used in the construction chosen to introduce correlation across the marginals.

This kind of construction, based on our invariance principle, applies not only to the gamma process but to many other kinds of processes as well. Section 3, for example, introduces bivariate G-measures and bivariate variance gamma processes. Other important examples are bivariate and multivariate stable laws. A bivariate stable process with indices $0 < \alpha_1,\ \alpha_2 \leq 2$ is defined as follows. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sequence of i.i.d. random variables such that there is a sequence of positive constants $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ with

$$\left(a_n^{-1} \sum_{i=1}^{[nt]} X_i - c_n, b_n^{-1} \sum_{i=1}^{[nt]} Y_i - d_n \right) \xrightarrow{d} (S_1(t), S_2(t))$$

in $D[0, 1] \times D[0, 1]$ with respect to Skorohod topology, for some sequence of real constants $(c_n)_{n \geq 1}$ and $(d_n)_{n \geq 1}$ (see Resnick and Greenwood (1979) for details). The limiting process given above is refered to as a bivariate stable process. Such processes are part of the class of multivariate processes generated by our technique.

The invariance principle has other interesting applications as well. For example, Section 4 uses the invariance principle to derive finite-dimensional approximating processes. We start by focusing on the univariate case and demonstrate

some useful approximations to Dirichlet processes and G-measures. We then derive a simple finite-dimensional approximation to the bivariate gamma process. When normalized this approximates a Dirichlet bi-measure. Bivariate and multivariate Dirichlet process modeling is an active area of research in Bayesian nonparametric statistics, and such constructions should be of great interest (see Bulla, Muliere, and Walker (2007), Ghosh, Hjort, Messan and Ramamoorthi (2006) and Walker and Muliere (2003) for related examples). In Section 5 we consider such a type of correlated process as a prior in a Bayesian image enhancement example.

Remark 1. Proofs for all results can be found in the online version of the paper available at `http://www.stat.sinica.edu.tw/statistica`.

2. A Scale Invariance Principle

The first result captures the essence of the scale invariance principle and is used repeatedly throughout the paper.

Theorem 1. *Let $\{U_i\}$ and $\{V_i\}$ be mutually independent sequences of i.i.d. positive random variables on $(0, \infty]$, with $0 < h = [\mathrm{E}\,(U_1^{-1})]^{-1} < \infty$ and $0 < g = [\mathrm{E}\,(V_1^{-1})]^{-1} < \infty$. If $N(\cdot)$ is a nonnegative strictly decreasing function defined on $(0, \infty)$ such that*

$$\int_0^\infty N^{-1}(x)dx < \infty, \tag{2.1}$$

then one has the following over the space of point processes.

(i) $\sum_{i=1}^\infty \varepsilon_{(N^{-1}(\Gamma_i U_i), N^{-1}(\Gamma_i V_i))}(\cdot)$ *is a Poisson random measure with mean measure* Π*, where*

$$\Pi((a, \infty) \times (b, \infty)) = \mathrm{E}\left(\frac{N(a)}{U_1} \bigwedge \frac{N(b)}{V_1}\right), \quad a, b > 0.$$

(ii) $\sum_{i=1}^\infty \varepsilon_{N^{-1}(\Gamma_i V_i)}(\cdot) \stackrel{\mathscr{D}}{=} \sum_{i=1}^\infty \varepsilon_{N^{-1}(\Gamma_i h)}(\cdot).$

(iii) *If $\{X_i\}$ is mutually independent of $\{V_i\}$,*

$$\sum_{i=1}^\infty \varepsilon_{(N^{-1}(\Gamma_i V_i), X_i)}(\cdot) \stackrel{\mathscr{D}}{=} \sum_{i=1}^\infty \varepsilon_{(N^{-1}(\Gamma_i h), X_i)}(\cdot),$$

$$\sum_{i=1}^\infty N^{-1}(\Gamma_i V_i)\varepsilon_{X_i}(\cdot) \stackrel{\mathscr{D}}{=} \sum_{i=1}^\infty N^{-1}(\Gamma_i h)\varepsilon_{X_i}(\cdot).$$

2.1. Remarks

A few remarks about Theorem 1 are in order.

1. The convergence criteria (2.1) can be easily verified for the class of G-measures, as follows. Replace N by the Lévy measure for a $\mathscr{G}(\alpha, \delta, \theta)$-distribution,

$$M_{\alpha,\delta,\theta}(x) = \int_x^\infty \frac{\delta}{\Gamma(1-\alpha)} u^{-\alpha-1} \exp(-\theta u) du.$$

To verify (2.1) observe that

$$\int_0^\infty M_{\alpha,\delta,\theta}^{-1}(x) dx \leq \int_0^\infty u M_{\alpha,\delta,\theta}(du)$$
$$= \frac{\delta}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha} \exp(-\theta u) du.$$

The integral on the right-hand side is finite for any $0 < \alpha \leq 1$ and $\theta \geq 0$. If $\alpha \leq 0$ and $\theta > 0$, then the integral is again finite, equaling $\Gamma(1-\alpha)\theta^{-1-\alpha}$. These conditions are satisfied by the constraints on α and θ (recall (1.3)).

2. The convergence of $\sum_{i=1}^\infty N^{-1}(\Gamma_i V_i)$ is guaranteed under (2.1) if $h^{-1} < \infty$. This is seen by noting that $\Gamma_i/i \overset{\text{a.s.}}{\to} 1$ implies $\sum_{i=1}^\infty N^{-1}(\Gamma_i V_i) < \infty$ if $\sum_{i=1}^\infty N^{-1}(i V_i) < \infty$. By the Kolmogorov Three Series Theorem, the latter sum converges because

$$\sum_{i=1}^\infty \mathbb{P}\left\{ N^{-1}(i V_i) > 1 \right\} = \sum_{i=1}^\infty \mathbb{P}\left\{ N(1) V_1^{-1} > i \right\} \leq \mathrm{E}\left(N(1) V_1^{-1} \right) < \infty.$$

Note that the second equality follows because N^{-1} is decreasing. For the second condition of the Three Series Theorem, observe that

$$\sum_{i=1}^\infty \mathrm{E}\left(N^{-1}(i V_i) \right) = \mathrm{E}\left(\sum_{i=1}^\infty N^{-1}(i V_1) \right)$$
$$\leq \mathrm{E}\left(\int_0^\infty N^{-1}(x V_1) dx \right)$$
$$= \mathrm{E}(V_1^{-1}) \int_0^\infty N^{-1}(x) dx < \infty.$$

For the last condition,

$$\sum_{i=1}^\infty \mathrm{Var}\left(N^{-1}(i V_i) \right) \leq \sum_{i=1}^\infty \mathrm{E}\left(N^{-1}(i V_i) \right)^2 < \infty.$$

The sum on the right-hand side is bounded because $(N^{-1}(i V_i))^2$ is eventually smaller than $N^{-1}(i V_i)$.

3. An alternate series representation for the gamma process is obtained using part (iii) of Theorem 1. It follows that

$$G(\cdot) = \sum_{i=1}^{\infty} N_{\alpha,\beta}^{-1}(\Gamma_i V_i) \varepsilon_{X_i}(\cdot) \overset{\mathscr{D}}{=} \sum_{i=1}^{\infty} N_{\alpha/h,\beta}^{-1}(\Gamma_i) \varepsilon_{X_i}(\cdot).$$

We call G is a gamma process with scale parameter β and shape measure $\kappa(\cdot) = \alpha P_0(\cdot)/h$ (recall P_0 is the distribution for X_1).

4. Let $N(x) = x^{-\alpha}$ for $0 < \alpha < 1$. This is the Lévy measure for a stable law with index α, and is another special case of a G-measure (here $\theta = 0$). Suppose the sequence $\{V_i\}$ is chosen so that $h = \left[\mathrm{E}\left(V_1^{\alpha}\right)\right]^{-1} < \infty$. Then by part (iii) of Theorem 1,

$$\sum_{i=1}^{\infty} V_i \Gamma_i^{-1/\alpha} = \sum_{i=1}^{\infty} N^{-1}(V_i^{-\alpha} \Gamma_i)$$

$$\overset{\mathscr{D}}{=} \sum_{i=1}^{\infty} N^{-1}(\Gamma_i h) = \left[\mathrm{E}\left(V_1^{\alpha}\right)\right]^{1/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha}.$$

This is the well known series representation for a positive stable law (LePage (1981) and Rosinski (2001)).

3. Bivariate Processes

We are now in a position to present a new series representation for a bivariate gamma process.

Corollary 1. *Let $\{(V_{i,1}, V_{i,2})\}$ be an i.i.d. sequence of random vectors on $(0, \infty] \times (0, \infty]$ with joint distribution $H(\cdot, \cdot)$, constructed so as to be mutually independent of $\{\Gamma_i\}$ and $\{X_i\}$. Suppose that $0 < h_1 = [\mathrm{E}(V_{1,1}^{-1})]^{-1} < \infty$ and $0 < h_2 = [\mathrm{E}(V_{1,2}^{-1})]^{-1} < \infty$. Then the bivariate process*

$$\mathbf{G}(\cdot) = (G_1(\cdot), G_2(\cdot)) = \sum_{i=1}^{\infty} \left(N_{\alpha_1,\beta_1}^{-1}(\Gamma_i V_{i,1}), N_{\alpha_2,\beta_2}^{-1}(\Gamma_i V_{i,2}) \right) \varepsilon_{X_i}(\cdot)$$

is infinitely divisible; its marginals G_1 and G_2 are gamma processes with shape measures $\alpha_1 P_0/h_1$ and $\alpha_2 P_0/h_2$, and scale parameters β_1 and β_2, respectively.

We call $\mathbf{G}$ a bivariate gamma process with shape bi-measure $(\alpha_1 h_1^{-1}, \alpha_2 h_2^{-1}) P_0$, scale parameters (β_1, β_2), and dependence structure H.

3.1. Independent increments

It is not difficult to see that the presence of variables $\{(V_{i,1}, V_{i,2})\}$ in $\mathbf{G}$ must

induce some type of correlation structure between the marginal distributions. Here we explicitly quantify what this dependence is in terms of the joint distribution H on $(V_{i,1}, V_{i,2})$. Another question we also consider the conditions on H that are required for G_1 and G_2 to be independent.

We consider the second question first. It turns out that independence imposes a severe restriction on the support for H, namely that H must satisfy

$$H\Big(\{(x,\infty) : x > 0\} \cup \{(\infty,y) : y > 0\}\Big) = 1. \qquad (3.1)$$

This condition for independent marginals is similar to that seen for bivariate stable laws. See Section 2 of Samorodnitsky and Taqqu (1994). In fact H here plays the same role as that of a spectral measure for stable laws. The condition (3.1) means that the processes can never jump together and is equivalent to Proposition 5.3 and Example 5.3 of Cont and Tankov (2004).

3.2. Correlation properties

We quantify the correlation between $G_1(\mathscr{X})$ and $G_2(\mathscr{X})$ in terms of H. Let $M(t_1, t_2)$ be the joint moment generating function of $(G_1(\mathscr{X}), G_2(\mathscr{X}))$. Using the recursive technique in Theorem 1 of Banjevic, Ishwaran and Zarepour (2002), one can show that

$$\text{Cov}\left(G_1(\mathscr{X}), G_2(\mathscr{X})\right) = \frac{\partial^2 \log M(0,0)}{\partial t_1 \partial t_2}$$
$$= \iint_0^\infty N_{\alpha_1,\beta_1}^{-1}(sv_1) N_{\alpha_2,\beta_2}^{-1}(sv_2) ds H(dv_1, dv_2).$$

Therefore $\text{Cov}\left(G_1(\mathscr{X}), G_2(\mathscr{X})\right) \geq 0$, with equality holding if and only if H satisfies (3.1). Hence, if G_1 is independent of G_2, we must have $\text{Cov}\left(G_1(\mathscr{X}), G_2(\mathscr{X})\right) = 0$.

3.3. Bivariate G-measures

A bivariate G-measure can be defined in a similar fashion as in Corollary 1. One simply replaces the gamma Lévy measure, $N_{\alpha,\beta}(\cdot)$, with the Lévy measure of a $\mathscr{G}(\alpha, \delta, \theta)$-distribution, $M_{\alpha,\delta,\theta}(x)$. We call

$$\mathbf{G}(\cdot) = \sum_{i=1}^\infty \left(M_{\alpha,\delta,\theta}^{-1}(\Gamma_i V_{i,1}), M_{\alpha,\delta,\theta}^{-1}(\Gamma_i V_{i,2})\right) \varepsilon_{X_i}(\cdot)$$

a bivariate G-measure with shape bi-measure $(\delta h_1^{-1}, \delta h_2^{-1}) P_0(\cdot)$, index parameter α, and intensity parameter θ. Note that by Theorem 1 the marginals for the process are G-measures.

3.4. Bivariate variance gamma processes

Assuming no drift term, the variance gamma process assumes that the change in the log-price of a stock is

$$\Delta W(G(t)) \overset{\mathscr{D}}{=} \Delta G_1(t) - \Delta G_2(t),$$

where $G_1(t)$ and $G_2(t)$ are independent gamma processes with common shape and scale parameters (Madan, Carr and Chang (1998)). Notice that the variance gamma process can be written as

$$W(G(\cdot)) = \sum_{i=1}^{\infty} N_{\alpha,\beta}^{-1}(\Gamma_i)\varepsilon_{X_i}(\cdot) - \sum_{i=1}^{\infty} N_{\alpha,\beta}^{-1}(\Gamma_i')\varepsilon_{X_i'}(\cdot),$$

where $\Gamma_i' \overset{\mathscr{D}}{=} \Gamma_i$ and $X_i' \overset{\mathscr{D}}{=} X_i$. All variables are mutually independent. A more efficient representation as a signed measure is also possible by exploiting the scale invariance principle.

Theorem 2. *Let $\{\Delta_i\}$ be an i.i.d. sequence independent of $\{\Gamma_i, X_i\}$ such that $\mathbb{P}\{\Delta_i = 1\} = 1/2 = 1 - \mathbb{P}\{\Delta_i = -1\}$. Then*

$$W(G(\cdot)) \overset{\mathscr{D}}{=} \sum_{i=1}^{\infty} \Delta_i N_{2\alpha,\beta}^{-1}(\Gamma_i)\varepsilon_{X_i}(\cdot). \tag{3.2}$$

Proof. The right-hand side of (3.2) is

$$\nu(\cdot) = \sum_{i=1}^{\infty} I\{\Delta_i = 1\} N_{2\alpha,\beta}^{-1}(\Gamma_i)\varepsilon_{X_i}(\cdot) - \sum_{i=1}^{\infty} I\{\Delta_i = -1\} N_{2\alpha,\beta}^{-1}(\Gamma_i)\varepsilon_{X_i}(\cdot).$$

Define variables $V_{i,1}^*$ and $V_{i,2}^*$ as

$$V_{i,1}^* = \begin{cases} 1 & \text{if } \Delta_i = 1 \\ \infty & \text{otherwise} \end{cases}, \qquad V_{i,2}^* = \begin{cases} 1 & \text{if } \Delta_i = -1 \\ \infty & \text{otherwise} \end{cases}.$$

Then,

$$\nu(\cdot) \overset{\mathscr{D}}{=} \sum_{i=1}^{\infty} N_{2\alpha,\beta}^{-1}(\Gamma_i V_{i,1}^*)\varepsilon_{X_i}(\cdot) - \sum_{i=1}^{\infty} N_{2\alpha,\beta}^{-1}(\Gamma_i V_{i,2}^*)\varepsilon_{X_i}(\cdot).$$

By the scale invariance principle of Theorem 1, each sum on the right-hand side is a gamma process with shape measure $\alpha P_0(\cdot)$ and scale parameter β. Using an argument as in Section 3.1, deduce that the two processes are also independent.

One can also define a bivariate variance gamma process along the lines of Collary 1. Using Theorem 2, we can write this compactly as a bivariate signed measure,

$$\mathbf{G}(\cdot) = \sum_{i=1}^{\infty} \Delta_i \Big(N_{2\alpha_1,\beta_1}^{-1}(\Gamma_i V_{i,1}), N_{2\alpha_2,\beta_2}^{-1}(\Gamma_i V_{i,2}) \Big) \varepsilon_{X_i}(\cdot).$$

We call **G** a bivariate variance gamma process.

3.5. Dirichlet bi-measures

Analogous to the Dirichlet process, a Dirichlet bi-measure is defined as a normalized bivariate gamma process. We call

$$(\mathscr{P}_1(\cdot), \mathscr{P}_2(\cdot)) = \left(\frac{G_1(\cdot)}{G_1(\mathscr{X})}, \frac{G_2(\cdot)}{G_2(\mathscr{X})} \right)$$

a Dirichlet bi-measure. Note that the marginals $\mathscr{P}_1$ and $\mathscr{P}_2$ are Dirichlet processes.

4. Limiting Processes

It is of practical interest to derive finite series approximations to the processes we have discussed. We begin by considering univariate processes, and later discuss extensions to the bivariate setting. The next theorem states a weak convergence result for partial sum processes involving $\mathscr{G}(\alpha, \theta, \delta)$ random variables. A useful byproduct of the theorem is a technique for simulating sample paths from certain types of G-measures as well as Dirichlet processes.

Theorem 3. *Let $\{Z_{i,n}\}$ be a sequence of i.i.d. variables such that*

$$n\mathbb{P}\{Z_{1,n} \in dx\} \xrightarrow{v} M_{\alpha,\delta,\theta}(dx) = \frac{\delta}{\Gamma(1-\alpha)} x^{-\alpha-1} \exp(-\theta x) dx, \qquad (4.1)$$

where $\xrightarrow{v}$ denotes vague convergence.

(i) *If $\{U_i\}$ is an i.i.d. sequence of uniform$[0,1]$ variables, independent of $\{\Gamma_i\}$,*

$$\sum_{i=1}^{n} \varepsilon_{(i/n, Z_{i,n})}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{(U_i, M_{\alpha,\delta,\theta}^{-1}(\Gamma_i))}(\cdot),$$

$$\sum_{i=1}^{[nt]} Z_{i,n} \xrightarrow{d} \sum_{i=1}^{\infty} M_{\alpha,\delta,\theta}^{-1}(\Gamma_i) I\{U_i \le t\}, \quad 0 \le t \le 1.$$

(ii) *If $Y_{i,n} = Z_{i,n} / \sum_{i=1}^{n} Z_{i,n}$ for $i = 1, \ldots, n$, $\sum_{i=1}^{n} \varepsilon_{Y_{i,n}}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{Y_i}(\cdot)$, where $Y_i = M_{\alpha,\delta,\theta}^{-1}(\Gamma_i) / \sum_{i=1}^{\infty} M_{\alpha,\delta,\theta}^{-1}(\Gamma_i)$. Furthermore, if $\{X_i\}$ is independent of $\{Y_{i,n}\}$, $\sum_{i=1}^{n} Y_{i,n} \varepsilon_{X_i}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} Y_i \varepsilon_{X_i}(\cdot)$.*

Remark 2. We use $\xrightarrow{d}$ to denote weak convergence with respect to the vague topology for point processes and weak convergence with respect to the Skorohod

topology for partial sum processes. From the proof of part (ii), it can be seen that the convergence also holds with respect to the uniform topology.

4.1. Dirichlet processes

Condition (4.1) is the key to exploiting Theorem 3. As our first example, we consider the case when $M_{\alpha,\delta,\theta}$ is the Lévy measure for a gamma random variable. The limits of interest in this case involve a gamma process and a Dirichlet process.

For convenience we work with the parameterization used by the Lévy measure $N_{\alpha,\beta}$. Condition (4.1) requires us to find variables $\{Z_{i,n}\}$ such that

$$n\mathbb{P}\{Z_{1,n} \in dx\} \xrightarrow{v} \alpha x^{-1}\exp\Big(-\frac{x}{\beta}\Big)dx.$$

This holds for variables satisfying

$$\mathbb{P}\{Z_{1,n} \in dx\} = \frac{\beta^{-\alpha/n}}{\Gamma(\alpha/n)}x^{\alpha/n-1}\exp\Big(-\frac{x}{\beta}\Big)dx,$$

because $n/\Gamma(\alpha/n) \to \alpha$. In other words, (4.1) is satisfied if $\{Z_{i,n}\}$ are i.i.d. gamma random variables with shape and scale parameters α/n and β. Notice that by part (i) of Theorem 3,

$$\sum_{i=1}^{n} Z_{i,n}\varepsilon_{X_i}(\cdot) \xrightarrow{d} G_{\alpha,\beta}(\cdot). \tag{4.2}$$

Simulating values from the finite-dimensional process on the left-hand side is straightforward. Thus, (4.2) is a handy method for approximating the gamma process (Ishwaran and James (2004)).

Part(ii) of Theorem 3 also has interesting implications. With $\{Z_{i,n}\}$ chosen as above, we have

$$\sum_{i=1}^{n} \frac{Z_{i,n}}{\sum_{i=1}^{n} Z_{i,n}}\varepsilon_{X_i}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} \frac{N_{\alpha,\beta}^{-1}(\Gamma_i)}{\sum_{i=1}^{\infty} N_{\alpha,\beta}^{-1}(\Gamma_i)}\varepsilon_{X_i}(\cdot).$$

The right-hand side is a normalized gamma process or, equivalently, a Dirichlet process. The left-hand side, therefore, provides a simple way to approximate such processes. See Ishwaran and Zarepour (2002) for more discussion on finite-dimensional approximations to the Dirichlet process.

4.2. G-measure limits

There are many ways to satisfy (4.1) in constructing a weak limit approximation to G-measures. All methods, however, naturally exploit the fact that the limit in (4.1) closely resembles a gamma density.

Here is just one way of taking advantage of this principle. We consider the case where $(\alpha, \delta, \theta) \in (-\infty, 0) \times (0, \infty) \times (0, \infty)$. Let

$$f_{1,n}(x) = x^{-2} I\left\{\frac{1}{2n} \leq x \leq \frac{1}{n}\right\},$$

$$f_{2,n}(x) = \frac{\theta^a x^{a-1}}{\Gamma(a)} \exp(-\theta x) I\left\{x > \frac{1}{n}\right\},$$

where $a = -\alpha > 0$. Note that $f_{2,n}$ is proportional to a truncated gamma density. Let

$$f_n(x) = \frac{f_{1,n}(x) + f_{2,n}(x)}{n + 1 - F(1/n)},$$

$$F(x) = \int_0^x \frac{\theta^a u^{a-1}}{\Gamma(a)} \exp(-\theta u) du.$$

It is easy to check that f_n is a density. We take $\{Z_{i,n}\}$ to be i.i.d. random variables drawn from f_n. Simulating these values is quite easy, one way is by expressing f_n as a mixture density. If $\mathbb{P}_{1,n}$ is the measure corresponding to the density $f_{1,n}/n$ and $\mathbb{P}_{2,n}$ is the measure corresponding to the truncated gamma density $f_{2,n}/(1 - F(1/n))$, then $\{Z_{i,n}\}$ are random variables with the mixture distribution

$$\mathbb{P}\{Z_{1,n} \in dx\} = f_n(x)dx = w_n \mathbb{P}_{1,n}(dx) + (1 - w_n)\mathbb{P}_{2,n}(dx),$$

where

$$w_n = \left(1 + \frac{1 - F(1/n)}{n}\right)^{-1}.$$

Condition (4.1) is satisfied for this choice of random variables because

$$n f_n(x)dx = \frac{f_{1,n}(x) + f_{2,n}(x)}{1 + O(1/n)} dx \to M_{\alpha,\delta,\theta}(dx),$$

where $\delta = \theta^a \Gamma(1 - \alpha)/\Gamma(a) = -\alpha\theta^{-\alpha}$.

4.3. Signed processes

Theorem 3 has an extension to signed processes. Replacing condition (4.1) with

$$n\mathbb{P}\{Z_{1,n} \in dx\} \xrightarrow{v} \frac{\delta}{\Gamma(1 - \alpha)} \left(\frac{1}{2} x^{-\alpha-1} \exp(-\theta x) I\{x > 0\}\right.$$
$$\left. + \frac{1}{2}|x|^{-\alpha-1} \exp(-\theta|x|) I\{x < 0\}\right) dx,$$

we get

$$\sum_{i=1}^{[nt]} Z_{i,n} \xrightarrow{d} \sum_{i=1}^{\infty} \Delta_i M_{\alpha,\delta,\theta}^{-1}(\Gamma_i) I\{U_i \leq t\},$$

where $\{\Delta_i\}$ is an i.i.d. sequence of Bernoulli$(1/2)$ random variables.

4.4. Bivariate gamma process limits

We extend the weak limit approximation given by (4.2) to bivariate gamma processes. The idea is to use a random shape parameter selected in such a manner to ensure that the limiting marginal distributions are gamma processes while at the same time introducing correlation between the marginal processes.

We first explain how the idea works in the univariate setting. Let $\{Z_{i,n}\}$ be i.i.d. random variables such that $Z_{i,n}$, conditioned on V_i, has a gamma distribution with shape parameter $\alpha V_i^{-1}/n$ and scale parameter β, where $\{V_i^{-1}\}$ are positive bounded i.i.d. random variables with $h = [\mathrm{E}\,(V_1^{-1})]^{-1} < \infty$ (the assumption of boundedness can be weakened). It follows that

$$\begin{aligned}
n\mathbb{P}\{Z_{1,n} \in dx\} &= n\mathrm{E}\left(\frac{\beta^{-\alpha V_1^{-1}/n}}{\Gamma(\alpha V_1^{-1}/n)} x^{\alpha V_1^{-1}/n-1} \exp\left(-\frac{x}{\beta}\right)\right) dx \\
&\to \mathrm{E}\,(V_1^{-1})\alpha x^{-1} \exp\left(-\frac{x}{\beta}\right) dx \\
&= N_{\alpha/h,\beta}(dx).
\end{aligned} \tag{4.3}$$

Hence, (4.1) is satisfied, and by by part (i) of Theorem 3 and the scale invariance principle of Theorem 1,

$$\sum_{i=1}^{n} Z_{i,n}\varepsilon_{X_i}(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} N_{\alpha,\beta}^{-1}(\Gamma_i V_i)\varepsilon_{X_i}(\cdot).$$

A limiting bivariate gamma process as in Theorem 1 can be constructed using the same idea. Let $\{(V_{i,1}^{-1}, V_{i,2}^{-1})\}$ be positive bounded i.i.d. random vectors with $h_j = [\mathrm{E}\,(V_{1,j}^{-1})]^{-1} < \infty$ for $j = 1, 2$. Let $F_{\alpha,\beta}$ denote the c.d.f. for a gamma random variable with shape and scale parameter α and β. Let $\{U_i\}$ be an i.i.d. sequence of uniform$[0, 1]$ random variables. We use the inverse probability transform to define gamma random variables as

$$Z_{i,n}^{(j)} = F_{\alpha_j^*,\beta_j}^{-1}(U_i), \quad i = 1, \ldots, n, \ j = 1, 2,$$

where $\alpha_j^* = V_{i,j}^{-1}\alpha_j/n$ (we suppress the dependence on i and n for notational clarity). Define the bivariate process

$$\mathbf{G}_n(\cdot) = \sum_{i=1}^{n} \left(Z_{i,n}^{(1)}, Z_{i,n}^{(2)}\right)\varepsilon_{X_i}(\cdot).$$

Then, $\mathbf{G}_n(\cdot)$ converges to a bivariate gamma process:

$$\mathbf{G}_n(\cdot) \xrightarrow{d} \sum_{i=1}^{\infty} \left(N_{\alpha_1,\beta_1}^{-1}(\Gamma_i V_{i,1}), N_{\alpha_2,\beta_2}^{-1}(\Gamma_i V_{i,2}) \right) \varepsilon_{X_i}(\cdot).$$

Convergence is proved using a similar approach as in Theorem 3. The key assumption to be verified is (c.f., part (i) of Theorem 1):

$$n\mathbb{P}\left\{ Z_{1,n}^{(1)} > a, Z_{1,n}^{(2)} > b \right\} \xrightarrow{v} \mathrm{E}\left(\frac{N_{\alpha_1,\beta_1}(a)}{V_{1,1}} \bigwedge \frac{N_{\alpha_2,\beta_2}(b)}{V_{1,2}} \right), \quad \text{for each} \quad a > 0, \ b > 0. \tag{4.4}$$

Notice that

$$
\begin{aligned}
\mathbb{P}\left\{ Z_{1,n}^{(1)} > a, Z_{1,n}^{(2)} > b \right\} &= \mathbb{P}\left\{ U_1 > F_{\alpha_1^*,\beta_1}(a), U_1 > F_{\alpha_2^*,\beta_2}(b) \right\} \\
&= \mathrm{E}\left(1 - F_{\alpha_1^*,\beta_1}(a) \bigvee F_{\alpha_2^*,\beta_2}(b) \right) \\
&= \mathrm{E}\left(\left(1 - F_{\alpha_1^*,\beta_1}(a) \right) \bigwedge \left(1 - F_{\alpha_2^*,\beta_2}(b) \right) \right).
\end{aligned}
$$

Using a similar argument as (4.3) deduce that

$$n\mathrm{E}\left(1 - F_{\alpha_j^*,\beta_j}(x) \Big| V_{1,j} \right) = n\mathbb{P}\left\{ Z_{1,n}^{(j)} > x \Big| V_{1,j} \right\} \to V_{1,j}^{-1} N_{\alpha_j,\beta_j}(x).$$

It follows, therefore, that (4.4) holds.

A similar weak limit approximation can be stated for multivariate gamma processes. Furthermore, by normalizing the underlying process we obtain a useful approximation to the multivariate Dirichlet process. For the bivariate case (Dirichlet bi-measure), this corresponds to

$$\frac{\mathbf{G}_n(\cdot)}{\mathbf{G}_n(\mathscr{X})} \xrightarrow{d} \sum_{i=1}^{\infty} \left(\frac{N_{\alpha_1,\beta_1}^{-1}(\Gamma_i V_{i,1})}{\sum_{i=1}^{\infty} N_{\alpha_1,\beta_1}^{-1}(\Gamma_i V_{i,1})}, \frac{N_{\alpha_2,\beta_2}^{-1}(\Gamma_i V_{i,2})}{\sum_{i=1}^{\infty} N_{\alpha_2,\beta_2}^{-1}(\Gamma_i V_{i,2})} \right) \varepsilon_{X_i}(\cdot).$$

5. Image Enhancement

In this section we present a Bayesian image enhancement application. At the heart of this illustration is the use of a multivariate gamma process $\mathbf{G}_n$ as a Bayesian prior. For our example we use the image depicted in Figure 1(a). We specifically focus on the zoomed in area, Figure 1(b), which has been blurred by adding Gaussian noise to the image; see Figure 1(c). The goal here is to enhance the blurred image using a Bayesian approach.

The data consists of paired values $(s, I(s))$. The value s is the two-dimensional spatial coordinate of a pixel on the image, while $I(s)$ is the intensity
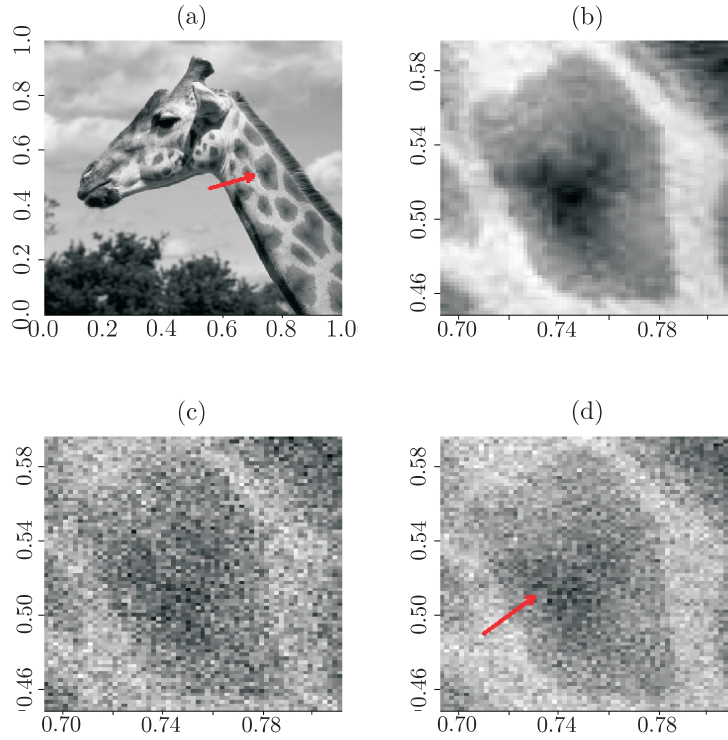
Figure 1. Image enhancement using correlated multivariate gamma prior: (a) original image; (b) zoom in of spot in (a) (indicated by arrow); (c) Image (b) after blurring; (d) Bayesian image enhancement of (c).

measure for the pixel. Here $I(s)$ is measured on a grey scale of 256 values, taking values in the range of $[0, 1]$. The larger the value, the whiter the pixel.

In order to enhance the image, we e stimate the intensity value from the blurred image by smoothing the intensity value for a given pixel using its eight adjacent neighbors. For a given pixel with location s and intensity $I(s)$, we use $s_1, \ldots, s_9$ to denote the spatial location of the neighborhood of s, and $I_1(s), \ldots, I_9(s)$ to denote the corresponding intensities. For convenience we set $s_1 = s$ and $I_1(s) = I(s)$. The estimated intensity at s is the smoothed value

$$\hat{I}(s) = \prod_{j=1}^{d} \int \phi_\tau \Big(s_j - v_{s,j}, I_j(s) \Big) \mu_j(dv_{s,j}),$$

where $d = 9$, $\phi_\tau(s, I) = I^{1/d} \exp(-\|s\|^2/\tau^2)$, and μ_j is an unknown positive measure defined over the space for s.

The estimator $\hat{I}(s)$ should take into account the correlation introduced by smoothing over adjacent pixels. For this reason we use a multivariate gamma process as in Section 4.4. Specifically, for the prior for $\mu = (\mu_1, \ldots, \mu_9)$, we use a 9-dimensional multivariate gamma process $\mathbf{G}_n$, where $Z_{i,n}^{(j)} = Z_i$ for $j = 1, \ldots, 9$, and Z_i are i.i.d. gamma variables with shape parameter α/n and scale parameter $\beta = 1$ (according to our notation of Section 4.4 this means $V_{i,j} = 1$). By the work in Sections 3.1 and 3.2 it is clear that $\mathbf{G}_n$ has correlated marginals. Indeed, this is a degenerate prior implying a very strong local dependence between pixels.

We estimate μ_j using a Gibbs sampling approach. For computational reasons it is necessary to augment the parameter space to include "missing values" $v_{s,j}$. One can think of $v_{s,j}$ as values sampled from μ_j. Let v be the vector composed of all $v_{s,j}$ values. The likelihood that we work with is

$$L(v) = \prod_s \prod_{j=1}^d \psi_{s,j}(v_{s,j}),$$

where $\psi_{s,j}(v_{s,j}) = \phi_\tau(s_j - v_{s,j}, I_j(s))$. The prior Q_n for (v, μ) is

$$Q_n(dv, d\mu) = \prod_s \Big(\mu_1(dv_{s,1}), \ldots, \mu_d(dv_{s,d}) \Big) \mathbf{G}_n(d\mu).$$

This assumes that $(v_{s,1}, \ldots, v_{s,d})$, given μ, has distribution μ, where μ has a correlated multivariate gamma process distribution.

5.1. Posterior characterization

Computations for this problem can be based on the blocked Gibbs sampling method (Ishwaran and Zarepour (2000); Ishwaran and James (2001)). To utilize this method we exploit the fact that a multivariate Dirichlet process is a normalized multivariate gamma process. This connection is summarized in the following theorem which characterizes the posterior distribution.

Theorem 4. *Let Q_n^* be the posterior for (v, μ) under the likelihood $L(v)$. If P_0 is non-atomic, then for any integrable function $g(v, \mu)$,*

$$\iint g(v, \mu) Q_n^*(dv, d\mu) = \iiint g(v^*, \mu) \frac{m_n(dK, dp, dX)}{\iiint m_n(dK, dp, dX)},$$

where

$$m_n(dK, dp, dX) = \left(\prod_s \prod_{j=1}^d \psi_{s,j}(v_{s,j}^*) \left\{ \sum_{i=1}^n p_i \varepsilon_i(dK_{s,j}) \right\} \right) \pi_n(dp) P_0^n(dX).$$

Here $v_{s,j}^* = X_{K_{s,j}}$ and $\pi_n(dp_1, \ldots, dp_n)$ is the density for a n-dimensional Dirichlet distribution with parameters $(\alpha/n, \ldots, \alpha/n)$.

5.2. Blocked Gibbs sampling

Theorem 4 is a template for applying the blocked Gibbs sampler (Ishwaran and Zarepour (2000) and Ishwaran and James (2001)). In this example, the blocked Gibbs sampler takes the problem of sampling (v, μ) from Q_n^* and turns it into the simpler problem of drawing values (K, p, X) from the density proportional to $m_n(dK, dp, dX)$.

To sample from the posterior, one simply draws conditional values (see for example Ishwaran and Zarepour (2000) and Ishwaran and James (2001, 2004)):

1. $(K|p, X)$;
2. $(p|K, X)$;
3. $(X|p, K)$.

Cycling through these three steps eventually yields a draw from the augmented density. By Theorem 4, a draw for v^* from the posterior can then be obtained by setting $v_{s,j}^* = X_{K_{s,j}}$. A posterior draw for μ is obtained by drawing μ^* according to $\mu^*(\cdot) = Z_0 \sum_{i=1}^n (p_1^*, \ldots, p_n^*) \varepsilon_{X_i^*}(\cdot)$, where p_i^* and X_i^* are Gibbs sampled values for p_i and X_i, and Z_0 is an independent gamma random variable with scale parameter $\beta = 1$ and shape parameter $\alpha + Sd$, where S is the total number of pixels.

5.3. Results

We applied the blocked Gibbs sampler using a 2,500 burn-in iteration. Posterior values were computed using 2,500 values collected after burn-in. A flat bivariate normal prior was used for P_0. We set $\alpha = 25$ and $n = 500$ for $\mathbf{G}_n$. The value of τ, which represents a bandwidth parameter, was set at one-half of the average distance between two pixels on the image. The intensity value for a pixel at s was estimated by averaging

$$I^*(s) = \prod_{j=1}^d \int \phi_\tau \Big(s_j - v_{s,j}, I_j(s) \Big) \mu_j^*(dv_{s,j})$$

$$= \prod_{j=1}^d \left(Z_0 \sum_{i=1}^n p_{i,j}^* \phi_\tau \Big(s_j - X_i^*, I_j(s) \Big) \right)$$

over 2,500 posterior sampled values.

Figure 1(d) records the posterior estimate and, as can be seen, the image is noticeably improved. In particular, edges are more defined and the black spot in the center of the image (indicated by an arrow) is more pronounced. It is possible that the image could be even further improved by incorporating

information using a wider radius of adjacent pixels. Another approach might be to extend the definition for $Z_{i,n}^{(j)}$ to introduce a richer correlation structure. These and other enhancements are currently under investigation.

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Department of Quantitative Health Sciences, Cleveland Clinic, Desk JJN3-01 9500 Euclid Ave. Cleveland, OH 44195, U.S.A.

E-mail: hemant.ishwaran@gmail.com

Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Avenue, Ottawa, Ontario, K1N 6N5, Canada.

E-mail: zarepour@uottawa.ca